Solving for binary inspiral dynamics using renormalization group methods

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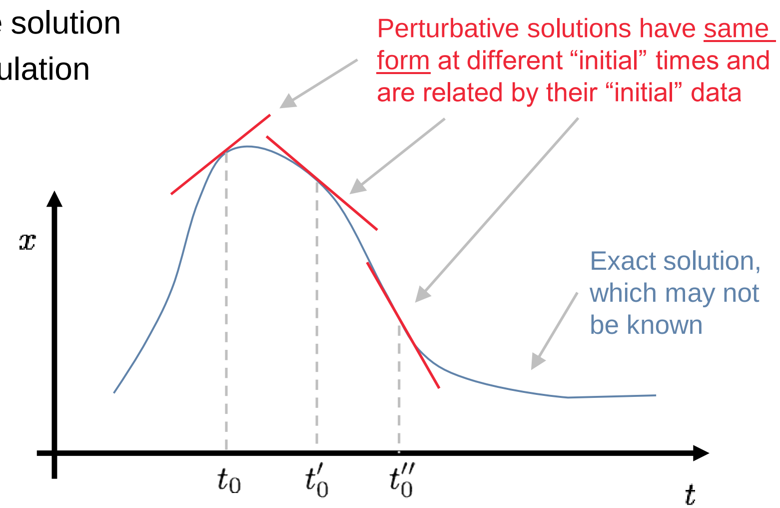
Motivation

- Solving equations of motion for compact binary inspirals is important but has challenges
  - Must be achieved using numerical methods, which is a bottleneck for gravitational wave data analysis applications
  - Important phase errors over many thousands of orbits (e.g., in LIGO's bandwidth) can be caused by inaccurately capturing the effects of very weak nonconservative forces
  - Often can involve using high-order adaptive solvers to provide sufficiently accurate numerical solutions over a very large number of orbits
  - Perturbative solutions exhibit secular behavior making result invalid over short times
  - At least one of these issues are often encountered in solving other types of nonconservative equations of motion
- Most analytical methods for gravitational wave source problems are based on orbit-averaging/adiabatic approximations
  - Advantages:
    - Simpler equations to solve
    - Often provides useful qualitative understanding of the system's physical tendencies
  - Disadvantages:
    - Ambiguity about timescale to use for averaging: Period is associated with mean, eccentric, or true anomalies? [see Pound & Poisson (2008)]
    - Not a systematic procedure
    - What are the errors of the resulting approximate solutions?
    - Lose real-time phase information
    - Tend to be less useful as a system becomes more complicated (e.g., precession) [see Chatziioannou et al (2016) for recent progress]

Dynamical Renormalization Group

Overview

- Background:
  - Introduced as a method for solving ODE's by Chen, Goldenfeld, and Oono (1996)
  - Based on Renormalization Group Theory from high-energy and condensed matter physics
  - Encapsulates several other asymptotic methods of global analysis including:
    - Multiple-scale analysis
    - WKB theory
    - Boundary layer theory
  - Based on naive perturbation theory
  - Systematic
    - Provides a turn-the-crank method of finding globally valid approximate solutions
    - Provides a formal error estimate on the perturbative solution
    - Contains strong self-consistency checks of the calculation
- Basic idea
  - Time at which to build a perturbative solution is arbitrary
  - Perturbative solutions (at fixed order) at different times have the same form but different initial data parameters
  - These solutions are related to each other by "renormalization group flows" from one initial data set to another.
  - What gets renormalized? Initial data parameters.



$$x(t) = X_0 + V_0(t - t_0) + \mathcal{O}(t - t_0)^2$$

$$x(t) = X_0' + V_0'(t - t_0') + \mathcal{O}(t - t_0)^2 \qquad t_0' = t_0 + \delta t \implies X_0' \approx X_0 + V_0 \delta t\ , \quad V_0' \approx V_0$$

Dynamical Renormalization Group

The algorithm

- Write down the equations of motion
- Write down a background solution around which to perturb
  - This solution is written in terms of "bare" parameters (i.e., $R_B(t_0)$), which implicitly depend upon the initial time t_0, away from which we flow.
- Use this background to calculate perturbatively the solution to equations of motion.
  - The perturbation will in general have secular "divergences" (i.e., terms that grow as $(t-t_0)$).
- Take this solution and write the bare parameters as renormalized parameters (i.e., $R_R(\tau)$) plus "counter-terms".
  - Counter-terms will be proportional to $(\tau-t_0)^p$ and are chosen to eliminate the t_0 dependence of the aforementioned solution.
  - τ is known as the "subtraction point" or "renormalization scale."
  - This step yields the "renormalized" perturbative solution.
  - Renormalized solution must be independent of the choice of τ.
  - The solutions' explicit dependence on τ is cancelled by the implicit dependence of the renormalized parameters on τ.
  - Use this fact to derive a first-order differential equation (called the "renormalization group (RG) equation") for the renormalized parameter.
  - The right-hand side of the RG equation is the "beta (β) function."
- Solve the RG equations and set $\tau = t$, the observation time.
  - All of the secularly growing terms are resummed at this order in perturbation theory.

Quantum Field Theory		Dynamical Renormalization Group
p_α	$\longrightarrow$	t
Λ	$\longrightarrow$	t_0
μ	$\longrightarrow$	τ

Binary inspirals at leading post-Newtonian order

Equations of motion

- 0PN equations of motion in polar coordinates (motion occurs in a plane for all time)

$$\ddot{r} - r\omega^2 = -\frac{M}{r^2} + \frac{64M^3\nu}{15r^4}\dot{r} + \frac{16M^2\nu}{5r^3}\dot{r}^3 + \frac{16M^2\nu}{5r}\dot{r}\omega^2$$

$$r\dot{\omega} + 2\dot{r}\omega = -\frac{24M^3\nu}{5r^3}\omega - \frac{8M^2\nu}{5r^2}\dot{r}^2\omega - \frac{8M^2\nu}{5}\omega^3$$

- Radiation reaction from gravitational wave emission causes orbit to depart from a background orbit
  - For definiteness, consider a background circular orbit with a Keplerian angular frequency

$$\Omega_B^2 = \frac{M}{R_B^3}$$

  - Perturbed orbit is described by:

$$r(t) = R_B + \delta r(t) \qquad \delta r/R_B \sim \mathcal{O}(v^5)$$

$$\omega(t) = \Omega_B + \delta\omega(t) \qquad \delta\omega/\Omega_B \sim \mathcal{O}(v^5) \qquad v \sim R_B\Omega_B$$

- Expand equations of motion to first order in perturbations off of background orbit

$$\delta\ddot{r}(t) - 3\Omega_B^2\delta r(t) - 2R_B\Omega_B\delta\omega(t) = 0$$

$$R_B\delta\dot{\omega}(t) + 2\Omega_B\delta\dot{r}(t) = -\frac{32\nu}{5}R_B^6\Omega_B^7$$

General solution

- General solution is parameterized by four numbers (the bare parameters, “B”)

$$r(t) = R_B - \frac{64\nu}{5}\Omega_B^6 R_B^6 (t-t_0) + \frac{64\nu}{5}\Omega_B^5 R_B^6 \sin\Omega_B(t-t_0) + A_B \sin\left(\Omega_B(t-t_0)+\Phi_B\right)$$

$$\omega(t) = \Omega_B + \frac{96\nu}{5} R_B^5 \Omega_B^7 (t-t_0) - \frac{128\nu}{5} R_B^5 \Omega_B^6 \sin\Omega_B(t-t_0) - \frac{2\Omega_B A_B}{R_B}\sin\left(\Omega_B(t-t_0)+\Phi_B\right)$$

- Can shift some bare parameters to remove non-secular sinusoids using trig identities

$$A_B \to A_B - \frac{64\nu}{5} R_B^6 \Omega_B^5 \cos\Phi_B$$

$$\Phi_B \to \Phi_B + \frac{64}{5}\frac{\nu R_B^6 \Omega_B^5}{A_B}\sin\Phi_B$$

- This results in the following general perturbed solution:

$$r(t) = R_B - \frac{64\nu}{5} R_B^6 \Omega_B^6 (t-t_0) + A_B \sin\left(\Omega_B(t-t_0)+\Phi_B\right)$$

$$\omega(t) = \Omega_B + \frac{96\nu}{5} R_B^5 \Omega_B^7 (t-t_0) - \frac{2\Omega_B A_B}{R_B}\sin\left(\Omega_B(t-t_0)+\Phi_B\right)$$

$$\phi(t) = \Phi_B + \Omega_B(t-t_0) + \frac{48\nu}{5} R_B^5 \Omega_B^7 (t-t_0)^2 + \frac{2A_B}{R_B}\cos\left(\Omega_B(t-t_0)+\Phi_B\right)$$

- Two types of perturbations off of background orbit
  - Secular terms (grow linearly with time and eventually invalidate the perturbative solution)

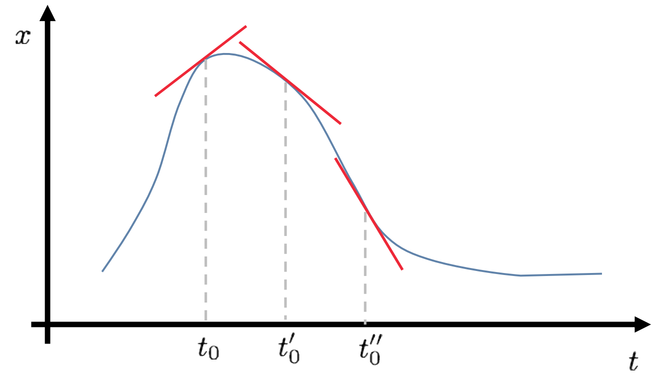
$$t - t_0 \sim \frac{1}{\nu\Omega_B^6 R_B^5} \sim \frac{1}{\nu v_B^5 \Omega_B} \implies \epsilon = v^5 \nu \Omega (t-t_0) \ll 1$$

(expansion parameter for DRG)

  - Non-secular terms (bounded in time)

Renormalization

- Renormalize the initial data parameters
  - Parameters depend implicitly on initial time
  - Write a bare ("B") parameter as a renormalized ("R") parameter plus a "counter-term"
  - Use counter-terms to absorb secular divergences



$$
\begin{aligned}
R_B(t_0) &= R_R(\tau) + \boxed{\delta_R(\tau, t_0)} \\
\Phi_B(t_0) &= \Phi_R(\tau) + \boxed{\delta_\Phi(\tau, t_0)} \\
\Omega_B(t_0) &= \Omega_R(\tau) + \boxed{\delta_\Omega(\tau, t_0)} \\
A_B(t_0) &= A_R(\tau) + \boxed{\delta_A(\tau, t_0)}
\end{aligned}
$$

Counter-terms

$$
\begin{aligned}
\delta_\Phi &= \mathcal{O}(1) \\
\delta_R, \delta_\Omega &= \mathcal{O}(\epsilon) \\
\delta_A &= \mathcal{O}(\epsilon^2)
\end{aligned}
$$

(A is already $O(\epsilon)$)

- Write perturbative solutions in terms of renormalized parameters
  - Drop higher order terms in ϵ for consistency

$$
r(t) = R_R + \delta_R - \frac{64\nu}{5} R_R^6 \Omega_R^6 (t - t_0) + A_R \sin\left((t - t_0)\Omega_R + \Phi_R + \delta_\Phi\right)
$$

$$
\omega(t) = \Omega_R + \delta_\Omega + \frac{96\nu}{5} R_R^5 \Omega_R^7 (t - t_0) - \frac{2\Omega_R A_R}{R_R} \sin\left((t - t_0)\Omega_R + \Phi_R + \delta_\Phi\right)
$$

$$
\phi(t) = \Phi_R + \delta_\Phi + (t - t_0)(\Omega_R + \delta_\Omega) + \frac{48\nu}{5} R_R^5 \Omega_R^7 (t - t_0)^2 + \frac{2A_R}{R_R} \cos\left((t - t_0)\Omega_R + \Phi_R + \delta_\Phi\right)
$$

- Introduce the subtraction point/renormalization scale τ through $t\text{-}t_0 = (t\text{-}\tau)+(\tau\text{-}t_0)$
- Choose counter-terms to remove $(\tau\text{-}t_0)$ dependencies

$$
\begin{aligned}
r(t) &= R_R + \boxed{\delta_R} - \frac{64\nu}{5} R_R^6 \Omega_R^6 (t-\tau) \boxed{- \frac{64\nu}{5} R_R^6 \Omega_R^6 (\tau - t_0)} + A_R \sin\left((t-\tau)\Omega_R + \boxed{(\tau - t_0)\Omega_R} + \Phi_R + \boxed{\delta_\Phi}\right) \\
\omega(t) &= \Omega_R + \boxed{\delta_\Omega} + \frac{96\nu}{5} R_R^5 \Omega_R^7 (t-\tau) + \boxed{\frac{96\nu}{5} R_R^5 \Omega_R^7 (\tau - t_0)} - \frac{2\Omega_R A_R}{R_R} \sin\left((t-\tau)\Omega_R + \boxed{(\tau - t_0)\Omega_R} + \Phi_R + \boxed{\delta_\Phi}\right) \\
\phi(t) &= \Phi_R + \boxed{\delta_\Phi} + (t-\tau)\Omega_R + \boxed{(\tau - t_0)\Omega_R} + \boxed{(t-\tau)\delta_\Omega} + \boxed{(\tau - t_0)\delta_\Omega} + \frac{48\nu}{5} R_R^5 \Omega_R^7 (t-\tau)^2 \\
&\quad + \boxed{\frac{96\nu}{5} R_R^5 \Omega_R^7 (t-\tau)(\tau - t_0)} + \boxed{\frac{48\nu}{5} R_R^5 \Omega_R^7 (\tau - t_0)^2} + \frac{2A_R}{R_R} \cos\left((t-\tau)\Omega_R + \boxed{(\tau - t_0)\Omega_R} + \Phi_R + \boxed{\delta_\Phi}\right)
\end{aligned}
$$

- Counter-terms through O(ϵ) are:

$$
\begin{aligned}
\delta_R(\tau, t_0) &= \frac{64\nu}{5} R_R^6 \Omega_R^6 (\tau - t_0) + \mathcal{O}(\epsilon^2) \\
\delta_\Omega(\tau, t_0) &= -\frac{96\nu}{5} R_R^5 \Omega_R^7 (\tau - t_0) + \mathcal{O}(\epsilon^2) \\
\delta_\Phi(\tau, t_0) &= -\Omega_R(\tau - t_0) + \frac{48\nu}{5} R_R^5 \Omega_R^7 (\tau - t_0)^2 + \mathcal{O}(\epsilon^2) \\
\delta_A(\tau, t_0) &= \mathcal{O}(\epsilon^2)
\end{aligned}
$$

Renormalization Group equations

- Recall: bare parameter = renormalized parameter + counter-term

$$
\begin{aligned}
R_B(t_0) &= R_R(\tau) + \frac{64\nu}{5} R_R^6 \Omega_R^6 (\tau - t_0) + \mathcal{O}(\epsilon^2 R_R) \\
\Omega_B(t_0) &= \Omega_R(\tau) - \frac{96\nu}{5} R_R^5 \Omega_R^7 (\tau - t_0) + \mathcal{O}(\epsilon^2 \Omega_R) \\
\Phi_B(t_0) &= \Phi_R(\tau) - \Omega_R(\tau - t_0) + \frac{48\nu}{5} R_R^5 \Omega_R^7 (\tau - t_0)^2 + \mathcal{O}(\epsilon^2) \\
A_B(t_0) &= A_R(\tau) + \mathcal{O}(\epsilon^2 A_R)
\end{aligned}
$$

- Note that the bare parameters are independent of τ
  - Differentiate the bare parameters with respect to τ and set the result to zero.
  - Solve for the derivative of the renormalized parameter.

$$
\begin{aligned}
\frac{dR_R}{d\tau} &= -\frac{64\nu}{5} R_R^6(\tau) \Omega_R^6(\tau) + \mathcal{O}(\epsilon^2 R_R \Omega_R) \\
\frac{d\Omega_R}{d\tau} &= \frac{96\nu}{5} R_R^5(\tau) \Omega_R^7(\tau) + \mathcal{O}(\epsilon^2 \Omega_R^2) \\
\frac{d\Phi_R}{d\tau} &= \Omega_R(\tau) \boxed{- \frac{d\Omega_R}{d\tau}(\tau - t_0)} \boxed{- \frac{96\nu}{5} R_R^5(\tau) \Omega_R^7(\tau)(\tau - t_0)} + \mathcal{O}(\epsilon^2 \Omega_R) \\
\frac{dA_R}{d\tau} &= \mathcal{O}(\epsilon^2 A_R \Omega_R)
\end{aligned}
$$

  - Secular pieces automatically cancel if the solution is renormalizable
    - Otherwise, secular divergences remain in renormalized parameters, which are supposed to be finite
    - This is a self-consistency check intrinsic to the DRG method

- Solve the RG equations to describe the "flow" from $\tau = t_i$ to $\tau = t$
  - Analytically, if possible
  - Numerically, otherwise (coupled first-order differential equations)

$$R_R(t) = \left(R_R^4(t_i) - \frac{256\nu}{5} M^3 (t - t_i)\right)^{1/4}$$

$$\Omega_R(t) = \Omega_R(t_i) \left(\frac{R_R(t_i)}{R_R(t)}\right)^{3/2}$$

$$\Phi_R(t) = \Phi_R(t_i) + \frac{1}{32\nu\Omega_R^5(t_i)R_R^5(t_i)} - \frac{1}{32\nu\Omega_R^5(t)R_R^5(t)}$$

$$A_R(t) = A_R(t_i)$$

- Substitute the RG solutions into the perturbative solutions and evaluate at $\tau = t$

$$r(t) = R_R(t) + A_R(t)\sin\Phi_R(t)$$

$$\omega(t) = \Omega_R(t) - \frac{2\Omega_R(t)A_R(t)}{R_R(t)}\sin\Phi_R(t)$$

$$\phi(t) = \Phi_R(t) + \frac{2A_R(t)}{R_R(t)}\cos\Phi_R(t)$$

Comments

- In analogy with quantum field theory calculations, first-order perturbative calculation is sometimes referred to as a "1-loop" calculation

- Solutions to RG equations resum secular divergences order-by-order in ϵ

$$R_R(t) = R_R(t_i)\left(1 - \frac{256\nu}{5}R_R^5(t_i)\Omega_R^6(t_i)(t-t_i)\right)^{1/4} + \mathcal{O}\big(R_R(t_i)v_R^5(t_i)\epsilon\big)$$

$$= R_R(t_i) - \frac{64\nu}{5}R_R^6(t_i)\Omega_R^6(t_i)(t-t_i) - \frac{6144\nu^2}{25}R_R^{11}(t_i)\Omega_R^{12}(t_i)(t-t_i)^2 + \mathcal{O}\big(R_R(t_i)\epsilon^3, R_R(t_i)v_R^5(t_i)\epsilon\big)$$

  - Third term is a secular divergence that appears at 2nd order but is already captured at first order by the resummation performed by DRG!

- Error estimates are naturally provided during the calculation

- DRG identifies (1-loop) invariants along the RG trajectory

$$R_R^3(t)\Omega_R^2(t) = \text{constant} = M \qquad \Phi_R(t) + \frac{1}{32\nu R_R^5(t)\Omega_R^5(t)} = \text{constant}$$

$$R_R^4(t)\left(1 + \frac{256\nu}{5}R_R^5(t)\Omega_R^5(t)\,t\right) = \text{constant} \qquad A_R(t) = \text{constant}$$

- Terms involving *(t-τ)(τ-t₀)* must be cancelled by pieces generated from counter-terms
  - Provides another self-consistency check of the calculation
  - Removal of such cross terms is important for the renormalizability of the perturbative solution

DRG to second order in ϵ: The 2-loop calculation

- Use same equations of motion but expanded to 2nd order in the perturbations.
- Find general solution to the 2nd order equations
- Shift bare parameters (i.e., initial data) to absorb redundant, finite pieces
  - These shifts have some freedom parameterized by μ.
  - Easiest to choose a "renormalization scheme" so as to keep the resulting 2-loop RG equations as simple as possible, which is equivalent to choosing μ to remove all the finite, t-dependent pieces in the expression for the 2nd order angular frequency solution.
- Renormalize initial data parameters to remove secular divergences.
  - For example:

$$
\begin{aligned}
r_{2-\mathrm{loop}}(t) = {} & \frac{1}{2}\frac{A_R^2}{R_R} - \frac{29\,696}{75}\nu^2 R_R^{11}\Omega_R^{10} \boxed{-\frac{6144}{25}\nu^2 R_R^{11}\Omega_R^{12}}\left[(t-\tau)^2 \boxed{-(\tau-t_0)^2}\right] \\
& - \frac{656}{15}\nu A_R R_R^5 \Omega_R^5 \cos\left(\Phi_R + \Omega_R(t-\tau)\right) + \frac{48}{5}\nu A_R R_R^5 \Omega_R^7 (t-\tau)^2 \cos\left(\Phi_R + \Omega_R(t-\tau)\right) \\
& + \frac{1}{2}\frac{A_R^2}{R_R}\cos\left(2\Phi_R + 2\Omega_R(t-\tau)\right) \boxed{-\frac{496}{15}\nu A_R R_R^5 \Omega_R^6}\left[(t-\tau) \boxed{+(\tau-t_0)}\right] \sin\left(\Phi_R + \Omega_R(t-\tau)\right) \\
& + \boxed{\delta_R^{v^{10}}} + \boxed{\delta_A^{v^{10}}} \sin\left(\Phi_R + (t-\tau)\Omega_R\right)
\end{aligned}
$$

  - Yields the counter-terms for R and A through 2-loops
  - Importantly, cross terms involving $(t-\tau)^p(\tau-t_0)^q$ automatically cancel with other terms containing lower-order counter-terms (self-consistency).

- At the end of the day, the counter-terms through 2-loops are

$$\delta_R = \frac{64\nu}{5} R_R^6 \Omega_R^6 (\tau - t_0) - \frac{6144}{25} \nu^2 R_R^{11} \Omega_R^{12} (\tau - t_0)^2 + \mathcal{O}(R_R \epsilon^3)$$

$$\delta_\Omega = -\frac{96\nu}{5} R_R^5 \Omega_R^7 (\tau - t_0) + \frac{16896}{25} \nu^2 R_R^{10} \Omega_R^{13} (\tau - t_0)^2 + \mathcal{O}(\Omega_R \epsilon^3)$$

$$\delta_A = \frac{496}{15} A_R \nu R_R^5 \Omega_R^6 (\tau - t_0) + \mathcal{O}(A_R \epsilon^3)$$

$$\begin{aligned}\delta_\Phi = & -\Omega_R(\tau - t_0) + \frac{48\nu}{5} R_R^5 \Omega_R^7 (\tau - t_0)^2 - \frac{5632}{25} \nu^2 R_R^{10} \Omega_R^{13} (\tau - t_0)^3 \\ & + \frac{504}{5} \nu A_R R_R^4 \Omega_R^5 \sin \Phi_B(t_0) - \frac{5}{4} \frac{A_R^2}{R_R^2} \sin 2\Phi_B(t_0) + \mathcal{O}(\epsilon^3)\end{aligned}$$

- RG equations for initial data parameters are

$$\frac{dR_R}{d\tau} = -\frac{64\nu}{5} R_R^6 \Omega_R^6$$

$$\frac{d\Omega_R}{d\tau} = \frac{96\nu}{5} R_R^5 \Omega_R^7$$

$$\frac{d\Phi_R}{d\tau} = \Omega_R$$

$$\frac{dA_R}{d\tau} = -\frac{496}{15} A_R \nu R_R^5 \Omega_R^6$$

  - A large number of cancellations happen to prevent secular terms from remaining in the RG equations (self-consistency)
  - RG equations and solutions for all renormalized quantities (except *A)* are same as at 1-loop

- Solution for A_R (= $e_R R_R$ where e_R is the orbit's small eccentricity) is

$$A_R(t) = A_R(t_i)\left(\frac{R_R(t)}{R_R(t_i)}\right)^{31/12} \Longrightarrow \quad e_R(t) \equiv \frac{A_R(t)}{R_R(t)} = e_R(t_i)\left(\frac{R_R(t)}{R_R(t_i)}\right)^{19/12}$$

  - Power of 19/12 accounts for the circularization of a compact binary inspiral
  - Matches the well-known expression of Peters (1964) in the limit of small orbital eccentricity.

- RG invariants are same as at 1-loop except for a 2-loop modification to A_R invariant:

$$A_R(t) = \text{constant} \quad \longrightarrow \quad e_R^{12}(t) R_R^{19}(t) = \text{constant}$$

- Full, resummed perturbative solution through 2nd order is:

$$\begin{aligned} r(t) = R_R(t)\Big[1 &+ e_R(t)\sin\Phi_R(t) + \frac{1}{2}e_R^2(t) - \frac{29\,696}{75}\nu^2 R_R^{10}(t)\Omega_R^{10}(t) \\ &- \frac{656}{15}\nu e_R(t) R_R^5(t)\Omega_R^5(t)\cos\Phi_R(t) + \frac{1}{2}e_R^2(t)\cos 2\Phi_R(t) + \mathcal{O}\big(v_R^{15}\Omega_R(t-t_i)\big)\Big] \end{aligned}$$

$$\omega(t) = \Omega_R(t)\Big[1 - 2e_R(t)\sin\Phi_R(t) + \frac{904}{15}\nu e_R(t) R_R^5(t)\Omega_R^5(t)\cos\Phi_R(t) - \frac{5}{2}e_R^2(t)\cos 2\Phi_R(t) + \mathcal{O}\big(v_R^{15}\Omega_R(t-t_i)\big)\Big]$$

$$\phi(t) = \Phi_R(t) + 2e_R(t)\cos\Phi_R(t) + \frac{504}{5}\nu e_R(t) R_R^5(t)\Omega_R^5(t)\sin\Phi_R(t) - \frac{5}{4}e_R^2(t)\sin 2\Phi_R(t) + \mathcal{O}\big(v_R^{15}\Omega_R(t-t_i)\big)$$

Binary inspirals at first post-Newtonian order

- Include 1PN radiation reaction force but 0PN potential (for demonstration)
- Following the same steps as for 0PN order, the 1-loop RG equations are

$$\frac{dR_R}{d\tau} = -\frac{64\nu}{5}R_R^6\Omega_R^6 - \frac{4\nu}{105}(336\nu - 3179)R_R^8\Omega_R^8$$
$$\frac{d\Omega_R}{d\tau} = \frac{96\nu}{5}R_R^5\Omega_R^7 + \frac{2\nu}{35}(336\nu - 3179)R_R^7\Omega_R^9$$
$$\frac{d\Phi_R}{d\tau} = \Omega_R\ , \quad \frac{dA_R}{d\tau} = 0$$

- Analytical solutions can be found when integrating these RG equations

$$-\frac{64\nu}{5}M^3(t-t_i) = \frac{1}{4}\left(R_R^4(t) - R_R^4(t_i)\right) + \frac{1}{3}\alpha M\left(R_R^3(t) - R_R^3(t_i)\right) + \frac{1}{2}\alpha^2 M^2\left(R_R^2(t) - R_R^2(t_i)\right)$$
$$+\alpha^3 M^3\left(R_R(t) - R_R(t_i)\right) + \alpha^4 M^4 \log\left(\frac{R_R(t) - \alpha M}{R_R(t_i) - \alpha M}\right)$$

$$\Omega_R(t) = \Omega_R(t_i)\left(\frac{R_R(t)}{R_R(t_i)}\right)^{3/2} = \frac{M^{1/2}}{R_R^{3/2}(t)}$$ (same as 0PN)

$$-\frac{32\nu}{5}M^{5/2}\left(\Phi_R(t) - \Phi_R(t_i)\right) = \frac{1}{5}\left(R_R^{5/2}(t) - R_R^{5/2}(t_i)\right) + \frac{1}{3}\alpha M\left(R_R^{3/2}(t) - R_R^{3/2}(t_i)\right) + \alpha^2 M^2\left(R_R^{1/2}(t) - R_R^{1/2}(t_i)\right)$$
$$-\alpha^{5/2}M^{5/2}\left[\tanh^{-1}\sqrt{\frac{R_R(t)}{\alpha M}} - \tanh^{-1}\sqrt{\frac{R_R(t_i)}{\alpha M}}\right]$$

$$\alpha = \frac{3179}{336} - \nu$$

Summary

- The Dynamical Renormalization Group method:
  - Is a systematic, turn-the-crank way to solve differential equations
  - Provides formal error estimates on the resulting globally valid approximate solutions
  - Generates perturbatively invariant quantities along a RG flow
  - Has built-in checks for self-consistency that can be used to verify correctness of the calculation
  - Subsumes other well-known global approximation methods including:
    - WKB
    - Multiple scale analysis
    - Boundary layer theory

- We've applied DRG to several problems, at varying levels of completion:
  - Damped harmonic oscillator (useful test ground for understanding the method in detail)
  - Nonspinning 0PN compact binary inspirals
  - Nonspinning 1PN compact binary inspirals (nearly complete)
  - Tidal dissipation of spinning, extended bodies in a binary (in progress)
  - Poynting-Robertson effect on motion of dust irradiated by a star (nearly complete)
  - Scalar self-force inspirals in a weak gravitational field

Future work (1)

- Apply DRG to precessing compact binary inspirals and other spinning systems
  - Can analytic solutions to the RG equations be found?
  - Provide a formal error estimate for the validity of the resummed perturbative solutions

- Do the RG invariants have symmetries associated with them?
  - Is there a “Noether’s Theorem” that relates continuous symmetry transformations to these quantities conserved throughout the RG flow (e.g., inspirals)?
  - Equal-mass and equal-spin-magnitude compact binary inspirals possess an inspiral-invariant quantity found empirically in Galley et al (2010): Is it derivable using DRG? Is there a similar expression more generally applicable?

$$\frac{2\hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{S}}_2 + (\hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{L}})(\hat{\boldsymbol{S}}_2 \cdot \hat{\boldsymbol{L}})}{\sqrt{5}}$$

- Can DRG be combined with numerical solutions of backgrounds?
  - If so, could be useful for resumming secular divergences encountered in numerical simulations of binary black holes for theories with corrections to general relativity [see Okounkova et al (2017)]
  - Could also be useful for calculating gravitational self-force inspirals [see Gralla & Wald (2008), Warburton et al (2012), Osburn et al (2016)]

Future work (2)

- Could DRG handle transient (orbital) resonances since averaging methods are not used? [e.g., see Flanagan & Hinderer (2012) for the breakdown of averaging]

- Other interesting possible applications include:
  - Exoplanet orbital evolutions
  - Binary inspirals/outspirals of not-so-compact bodies (e.g., mass-transferring stellar bodies)
  - Orbital mechanics of satellites

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